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The Polya algorithm in sequence spaces

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Abstract

In this paper we consider the problem of best approximation in $\ell_p(\mathbb{N})$, $1 < p \leqslant \infty$. If h_p, $1 < p < \infty$ denotes the best ℓ_p-approximation of the element $h \in \ell_1(\mathbb{N})$ from a finite-dimensional affine subspace K of $\ell_1(\mathbb{N})$, $h \notin K$, then $\lim_{p\to\infty} h_p = h^*_\infty$, where h^*_∞ is a best uniform approximation of h from K, the so-called strict uniform approximation. Our aim is to give a complete description of the rate of convergence of $\|h_p - h^*_\infty\|$ as $p \to \infty$ by proving that there are constants $L_1, L_2 > 0$ and $0 \leqslant a \leqslant 1$ such that

$$L_1 a^p \leqslant p\,\|h_p - h^*_\infty\| \leqslant L_2\, a^p,$$

for p large enough.

Keywords: Best approximation; Sequence spaces; Strict uniform approximation; Rate of convergence; Polya algorithm

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1. Introduction

For $1 \leqslant p < \infty$, we consider the usual $\ell_p(\mathbb{N})$ linear space of the sequences $x = \{x(j)\}_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\sum_{j\in\mathbb{N}} |x(j)|^p < \infty$, endowed with the p-norm

$$\|x\|_p = \Big(\sum_{j\in\mathbb{N}} |x(j)|^p \Big)^{1/p},$$

and the linear space $\ell_\infty(\mathbb{N})$ of the bounded sequences in $\mathbb{R}^{\mathbb{N}}$, with the uniform norm

$$\|x\| = \|x\|_\infty = \sup_{j\in\mathbb{N}}\{|x(j)|\}.$$

Note that, for all $p > 1$, $\ell_1(\mathbb{N}) \subset \ell_p(\mathbb{N}) \subset \ell_\infty(\mathbb{N})$. Moreover, if $x \in \ell_1(\mathbb{N})$ then $\|x\| = \max_{j\in\mathbb{N}} |x(j)|$ and

$$\|x\| \leqslant \|x\|_p \leqslant \|x\|_1. \tag{1}$$

Let $K \neq \emptyset$ be a subset of $\ell_1(\mathbb{N})$. For $h \in \ell_1(\mathbb{N}) \setminus K$ and $1 \leqslant p \leqslant \infty$ we say that $h_p \in K$ is a best ℓ_p-approximation of h from K if

$$\|h_p - h\|_p \leqslant \|f - h\|_p \quad \text{for all } f \in K.$$

If $p = \infty$ we will say that h_∞ is a best uniform approximation of h from K. If K is a finite-dimensional linear subspace of $\ell_1(\mathbb{N})$, then the existence of h_p is guaranteed. Moreover, there exists a unique best ℓ_p-approximation if $1 < p < \infty$. In general, the unicity of the best uniform approximation is not guaranteed. However, a unique "strict uniform approximation", h^*_∞, can be defined [6]. The strict uniform approximation satisfies the next property. Let H denote the set of the best uniform approximation of h from K. For every $g \in H$ we consider the sequence $\tau(g)$ whose coordinates are given by $|g(j) - h(j)|$ arranged in decreasing order. Then h^*_∞ is the only element in H which has $\tau(h^*_\infty)$ minimal in the lexicographic ordering. This definition of strict uniform approximation extends the one given by Rice [9] when K is a linear subspace of $\mathbb{R}^n$.

There are quite a few attempts to generalize Rice's definition of strict best approximation when K is a finite linear subspace of $C[a, b]$ or $C_0(T)$, the Banach space of all real-valued continuous functions f on T which vanish at infinity, endowed with the supremum norm, where T is a locally convex compact Hausdorff, (see e.g., [4,5,11,13]). The existence of the strict uniform approximation is related to the problem of constructing a continuous selection for the metric projection in $C_0(T)$. In [5] it is proved that the definition of the strict uniform approximation as the limit of the best L_p-approximation as $p \to \infty$ (if it exists) provides a natural continuous selection in $C_0(T)$. The discovery of the connection between the convergence of the Polya algorithm and the existence of continuous selection is due to Sommer [11,12].

When K is an affine subspace of $\mathbb{R}^n$, the convergence of h_p to h^*_∞ was proved in [1]. In this context, the first result about the rate of convergence of $\|h_p - h^*_\infty\|$ appears in [2]. In this paper it is showed that $p\,\|h_p - h^*_\infty\|$ is bounded. Subsequently, in [7] the authors established necessary and sufficient conditions on K to get that $p\,\|h_p - h^*_\infty\| \to 0$ as $p \to \infty$

and in [8] it is proved that there are constants $L_1, L_2 > 0$ and $0 \leqslant a \leqslant 1$, depending on *K*, such that

$$L_1 a^p \leqslant p \, \|h_p - h_\infty^*\| \leqslant L_2 a^p, \tag{2}$$

for all *p* large enough.

Throughout this paper *K* will be a finite-dimensional affine subspace of $\ell_1(\mathbb{N})$. We will assume that $h = 0$ and $0 \notin K$. This involves no loss of generality since all relevant properties are translation invariant. In this context, it is also known, [3,6], that $\lim_{p\to\infty} h_p = h_\infty^*$. In [3] the authors extend the result in [2] by proving that there exist $M > 0$ and $p_0 > 1$ such that

$$p \, \|h_p - h_\infty^*\| < M, \qquad \text{for all } p \geqslant p_0.$$

Our aim is to give a complete description of the rate of convergence of $\|h_p - h_\infty^*\|$ by generalizing the result (2) to our context of approximation in the space $\ell_1(\mathbb{N})$.

2. Notation and preliminary results

For $J \subseteq \mathbb{N}$ we denote $J^c = \mathbb{N} \setminus J$. Moreover for $v \in \ell_\infty(\mathbb{N})$ we define $\|v\|_J = \sup_{j\in J} |v(j)|$. Notice that $\|v\|_{\mathbb{N}} = \|v\|$. The italic letters *h*, *u*, *v*, *w* and *z* will be used to denote elements of $\ell_p(\mathbb{N})$.

Let *K* denote a finite-dimensional affine subspace of $\ell_1(\mathbb{N})$, $0 \notin K$, and h_∞^* (h_p, $1 < p < \infty$) be the strict uniform approximation (the best ℓ_p-approximation) of 0 from *K*. Thus, we can write $K = h_\infty^* + \mathcal{V}$, where $\mathcal{V}$ is a finite-dimensional linear subspace of $\ell_1(\mathbb{N})$ of dimension $m > 0$ ($\dim(\mathcal{V}) = m$). We will assume that $\|h_\infty^*\| = 1$ (this involves no loss of generality). Thus, from (1) and the definition of h_p, we obtain

$$\|h_p\| \leqslant \|h_p\|_p \leqslant \|h_\infty^*\|_p \leqslant \|h_\infty^*\|_1,$$

and also $\|h_p - h_\infty^*\| \leqslant \|h_p\| + \|h_\infty^*\| \leqslant 2\|h_\infty^*\|_1$.

Set $J_0 := \{j \in \mathbb{N} : h_\infty^*(j) = 0\}$ and denote

$$\mathcal{V}_0 = \{v \in \mathcal{V} : v(j) = 0, \text{ for all } j \in \mathbb{N} \setminus J_0\}. \tag{3}$$

By means of an inductive procedure we define $d_0 = 0$, and for $n \in \mathbb{N}$ such that $\mathbb{N} \neq \cup_{l=0}^{n-1} J_l$,

$$d_n = \|h_\infty^*\|_{\mathbb{N}\setminus \cup_{l=0}^{n-1} J_l} \quad \text{and} \quad J_n = \{j \in \mathbb{N} : |h_\infty^*(j)| = d_n\}.$$

If $\mathbb{N} = \cup_{l=0}^{n-1} J_l$ for some $n \in \mathbb{N}$, then we obtain a finite strictly decreasing sequence $\{d_l\}_{l=1}^{n-1}$ and a finite family of finite disjoint sets $\{J_l\}_{l=1}^{n-1}$. In this case we will put $d_n = 0$. In the opposite case, we obtain a strictly decreasing sequence $\{d_n\}_{n\in\mathbb{N}}$ and a denumerable family of finite disjoint sets $\{J_n\}_{n\in\mathbb{N}}$ such that $d_n \to 0$ and $\cup_{l=0}^{\infty} J_l = \mathbb{N}$.

For $v \notin \mathcal{V}_0$, define

$$r(v) = \min\{n \in \mathbb{N} : v(j) \neq 0, \text{ for some } j \in J_n\}. \tag{4}$$

If $h_{p_i} \neq h^*_\infty$, define

$$u_i = \frac{h_{p_i} - h^*_\infty}{\|h_{p_i} - h^*_\infty\|}. \tag{5}$$

Notice that $u_i \in \mathcal{V}$. Suppose that $h_{p_i} \neq h^*_\infty$ for infinitely many p_i, with $p_i \to \infty$ as $i \to \infty$. Since $\|u_i\| = 1$ and $\dim(\mathcal{V}) < \infty$, we can assume, taking a subsequence if necessary, that

$$\lim_{i\to\infty} u_i = u \in \mathcal{V} \tag{6}$$

with $\|u\| = 1$. In that follows, this vector u will play an important role.

The following is a well known result (see for instance [10]).

Theorem 1 (*Characterization of the best ℓ_p-approximation*). *A point $h_p \in \ell_p(\mathbb{N})$, $1 < p < \infty$, is the best ℓ_p-approximation of* 0 *from K if and only if*

$$\sum_{j\in\mathbb{N}} v(j)\left|h_p(j)\right|^{p-1}\operatorname{sgn}(h_p(j)) = 0, \quad \textit{for all } v \in \mathcal{V}. \tag{7}$$

Lemma 1. *For $p > 1$, let $z_p \in \ell_1(\mathbb{N})$ and $z \in \ell_1(\mathbb{N})$ be such that $\|z_p - z\| \to 0$ as $p \to \infty$. If $z \neq 0$, then for p sufficiently large,*

$$\sum_{j\in\mathbb{N}} z(j)z_p(j)|z_p(j)|^{p-2} > 0. \tag{8}$$

Proof. We can assume that $\|z\| = 1$. Write $z_p = z + w_p$, with $w_p \in \ell_1(\mathbb{N})$ and $\|w_p\| \to 0$ as $p \to \infty$. Let $S = \{j \in \mathbb{N} : |z(j)| = 1\}$. Put $\gamma = \|z\|_{S^c} < 1$ and choose $\delta > 0$ such that $\gamma + \delta < 1$.

For $\varepsilon = \min\{\delta, 1 - \gamma - \delta, (\gamma + \delta)/\|z\|_1\}$, there exists $p' > 1$ such that for $p > p'$, $\|w_p\| < \varepsilon$ and $\operatorname{sgn}(z_p(j)) = \operatorname{sgn}(z(j))$, for all $j \in S$.

If $j \in S$ and $p > p'$, then

$$|z_p(j)| = |z(j) + w_p(j)| \geqslant |z(j)| - |w_p(j)| \geqslant 1 - \|w_p\| > 1 - \varepsilon \geqslant \gamma + \delta.$$

On the other hand, if $j \in S^c$ and $p > p'$, then

$$|z_p(j)| = |z(j) + w_p(j)| \leqslant |z(j)| + |w_p(j)| \leqslant \gamma + \|w_p\| < \gamma + \varepsilon \leqslant \gamma + \delta.$$

So, taking into account that $z(j)z_p(j) \geqslant z(j)w_p(j)$, we have for $p > p'$,

$$\begin{aligned}\sum_{j\in\mathbb{N}} z(j)z_p(j)\left|z_p(j)\right|^{p-2} &= \sum_{j\in S} |z_p(j)|\left|z_p(j)\right|^{p-2} + \sum_{j\in S^c} z(j)z_p(j)\left|z_p(j)\right|^{p-2} \\ &\geqslant \sum_{j\in S} |z_p(j)|\left|z_p(j)\right|^{p-2} + \sum_{j\in S^c} z(j)w_p(j)\left|z_p(j)\right|^{p-2} \\ &> (\gamma+\delta)^{p-2}\Big((\gamma+\delta)\operatorname{card}(S) - \|w_p\|\sum_{j\in S^c}|z(j)|\Big) \\ &\geqslant (\gamma+\delta)^{p-2}\Big((\gamma+\delta) - \varepsilon\|z\|_1\Big) \geqslant 0. \qquad \square\end{aligned}$$

Remark 1. We stand out the fact that if the vector u is defined by (6), then $u \notin \mathcal{V}_0$ (see (3)). In other case, applying (7) with $v = u$ and $p = p_i$, we have

$$0 = \sum_{j\in\mathbb{N}} u(j)|h_{p_i}(j)|^{p_i-1}\mathrm{sgn}(h_{p_i}(j)) = \sum_{j\in J_0} u(j)|h_{p_i}(j)|^{p_i-1}\mathrm{sgn}(h_{p_i}(j)).$$

Dividing the above equation by $\|h_{p_i} - h^*_\infty\|^{p_i-1}$, we obtain (recall that $h^*_\infty(j) = 0$ for $j \in J_0$)

$$\sum_{j\in J_0} u(j)u_i(j)|u_i(j)|^{p_i-2} = 0.$$

From Lemma 1 we get a contradiction for i large enough.

Lemma 2. *Let the linear space $\mathcal{V}$ and the family $\{J_n\}$ be given as above. There exist an integer $r \geqslant 0$, a strictly increasing sequence of integers $\{\sigma(k)\}_{k=0}^r$, with $\sigma(0) = 0$, and linear subspaces $\mathcal{V}_k$ $(0 \leqslant k \leqslant r)$ of $\mathcal{V}$ such that, $\mathcal{V} = \oplus_{k=0}^r \mathcal{V}_k$ and if $1 \leqslant k \leqslant r$ and $v \in \mathcal{V}_k \setminus \{0\}$, then $v(j) = 0$ for all $j \in \cup_{l=1}^{\sigma(k)-1} J_l$ and $v(j) \neq 0$ for some $j \in J_{\sigma(k)}$.*

Proof. If $\dim(\mathcal{V}_0) = m$ (see (3)), we take $\mathcal{V}_0 = \mathcal{V}$ and $r = 0$. In other case, put $\sigma(0) = 0$ and suppose that we have constructed linear spaces $\mathcal{V}_k$ and the corresponding sequence $\{\sigma(k)\}$ for $k = 0, 1, \ldots, s$, with the property described above. If $\mathcal{V} = \oplus_{k=0}^s \mathcal{V}_k$, then by taking $r = s$ we conclude the proof. Otherwise, we write $\mathcal{V} = \left(\bigoplus_{l=0}^s \mathcal{V}_l\right) \oplus \mathcal{W}_s$, where $\mathcal{W}_s = \mathcal{V} \cap \left(\bigoplus_{l=0}^s \mathcal{V}_l\right)^\perp$. Set

$$\mathcal{U}_{s+1} = \{v \in \mathcal{W}_s : v(j) = 0, \quad \text{for all } j \in \cup_{l=1}^{\sigma(s)} J_l\}$$

and put $\sigma(s+1) := \min\{n \in \mathbb{N} : v(j) \neq 0$ for some $v \in \mathcal{U}_{s+1}$ and some $j \in J_n\}$. Note that $\sigma(s+1) > \sigma(s)$. Now, we take $\mathcal{V}_{s+1}$ as the linear space generated by a family $\mathcal{B}$ in $\mathcal{U}_{s+1}$ such that $\mathcal{B}|_{J_{\sigma(s+1)}}$ is a basis of $\mathcal{U}_{s+1}|_{J_{\sigma(s+1)}}$. Finally, observe that this involves a finite inductive procedure. □

Lemma 3. *If $v \notin \mathcal{V}_0$, then there are $j, j' \in J_{r(v)}$ (see (4)) such that $v(j)h^*_\infty(j) > 0$ and $v(j')h^*_\infty(j') < 0$.*

Proof. Suppose the contrary. We can assume that $\|v\| = 1$ and $v(j)h^*_\infty(j) \leqslant 0$ for all $j \in J_{r(v)}$ (if this is not the case, we take the vector $-v$ in place of v).

Fix a positive λ such that $\lambda < d_{r(v)} - d_{r(v)+1} \leqslant d_{r(v)}$ and consider the vector $\tilde{h} = h^*_\infty + \lambda v \in K$. If $r(v) > 1$ and $j \in \cup_{l=1}^{r(v)-1} J_l$, then $v(j) = 0$ and so $\tilde{h}(j) = h^*_\infty(j)$. On the other hand, for $j \in J_{r(v)}$, we have

$$|\tilde{h}(j)| = |h^*_\infty(j)| - \lambda|v(j)| \leqslant |h^*_\infty(j)|.$$

Notice that the last inequality is strict for some $j \in J_{r(v)}$. Finally, if $j \in \mathbb{N} \setminus \cup_{l=1}^{r(v)} J_l$, then

$$|\tilde{h}(j)| \leqslant |h^*_\infty(j)| + \lambda|v(j)| \leqslant d_{r(v)+1} + \lambda < d_{r(v)}.$$

So, the vector $\tilde{h}$ is a best uniform approximation of 0 from K that contradicts the definition of h^*_∞. □

3. Rate of convergence

The next Theorem was proved in [3]. However, we present here a simpler proof whose greater interest is that we do not use the fact that $h_p \to h^*_\infty$ to conclude that the sequence $p\|h_p - h^*_\infty\|$ is bounded. The convergence of h_p to h^*_∞ (as $p \to \infty$) follows as an immediate consequence of our result.

Theorem 2. *Let K be a finite-dimensional affine subspace of $\ell_1(\mathbb{N})$, $0 \notin K$. Let h_p, $1 < p < \infty$, denote the best ℓ_p-approximation of 0 from K and let h^*_∞ be the strict uniform approximation of 0 from K. Then there exist positive constants M and C such that, for $p > C$,*

$$p\,\|h_p - h^*_\infty\| \leqslant M.$$

Proof. It is sufficient to prove that $\liminf p_i\|h_{p_i} - h^*_\infty\| < \infty$ for every increasing sequence $p_i \in (1, \infty)$ such that $h_{p_i} \neq h^*_\infty$, $p_i \to \infty$, and $u_i \to u$ (see (5)) as $i \to \infty$. From Remark 1 we know that $u \notin \mathcal{V}_0$, thus the integer $r(u)$ is well defined (see (4)). From Lemma 3 there exists $j_0 \in J_{r(u)}$ such that $u(j_0)h^*_\infty(j_0) > 0$. Since

$$p_i\|h_{p_i} - h^*_\infty\| = p_i|h_{p_i}(j_0) - h^*_\infty(j_0)|\frac{\|h_{p_i} - h^*_\infty\|}{|h_{p_i}(j_0) - h^*_\infty(j_0)|},$$

and

$$\lim_{i\to\infty} \frac{\|h_{p_i} - h^*_\infty\|}{|h_{p_i}(j_0) - h^*_\infty(j_0)|} = \frac{1}{|u(j_0)|},$$

it is sufficient to prove that the sequence $p_i|h_{p_i}(j_0) - h^*_\infty(j_0)|$ is bounded.

We need some notation. For each i let Γ_i be the finite set of indices $j \in \mathbb{N}$ such that $|h_{p_i}(j)| > d_{r(u)}$ and $u(j) \neq 0$. Moreover, define

$$\gamma = (d_{r(u)} - d_{r(u)+1})/(2\|h^*_\infty\|_1).$$

Since $J_{r(u)}$ is a finite set and $u_i \to u$, there exists N such that, for $i > N$, $\|u_i - u\| < \gamma$ and $\operatorname{sgn}(u_i(j)) = \operatorname{sgn}(u(j))$, for all $j \in J_{r(u)}$ such that $u(j) \neq 0$.

If $i > N$, then $\operatorname{sgn}(h^*_\infty(j_0)) = \operatorname{sgn}(u(j_0)) = \operatorname{sgn}(u_i(j_0)) = \operatorname{sgn}(h_{p_i}(j_0) - h^*_\infty(j_0))$. Therefore $|h_{p_i}(j_0)| > |h^*_\infty(j_0)| = d_{r(u)}$ and $\Gamma_i \neq \emptyset$. Moreover, if $j \in \Gamma_i \cap J_{r(u)}$, then $\operatorname{sgn}(u(j)) = \operatorname{sgn}(u_i(j)) = \operatorname{sgn}(h_{p_i}(j) - h^*_\infty(j)) = \operatorname{sgn}(h_{p_i}(j))$.

On the other hand if $j \in \Gamma_i \cap J^c_{r(u)}$, then $\operatorname{sgn}(u_i(j)) = \operatorname{sgn}(u(j))$. Indeed, if $j \in \Gamma_i \cap J^c_{r(u)}$, then $|h^*_\infty(j)| \leqslant d_{r(u)+1}$ and then $|h_{p_i}(j) - h^*_\infty(j)| > d_{r(u)} - d_{r(u)+1}$. If $\operatorname{sgn}(u_i(j)) \neq \operatorname{sgn}(u(j))$, then

$$\|u_i - u\| \geqslant |u_i(j) - u(j)| \geqslant |u_i(j)| = \frac{|h_{p_i}(j) - h^*_\infty(j)|}{\|h_{p_i} - h^*_\infty\|} > \frac{d_{r(u)} - d_{r(u)+1}}{2\|h^*_\infty\|_1} = \gamma$$

and we arrive to a contradiction.

Now for $i > N$ (we use below (7) with $v = u$ and $p = p_i$)

$$
\begin{aligned}
&|u(j_0)|\left(1 + (p_i - 1)\frac{|h_{p_i}(j_0) - h^*_\infty(j_0)|}{d_{r(u)}}\right)\\
&\quad\leqslant |u(j_0)|\left(1 + \frac{h_{p_i}(j_0) - h^*_\infty(j_0)}{h^*_\infty(j_0)}\right)^{p_i-1} \leqslant |u(j_0)|\left|\frac{h_{p_i}(j_0)}{d_{r(u)}}\right|^{p_i-1}\\
&\quad\leqslant \sum_{j\in\Gamma_i} |u(j)|\left|\frac{h_{p_i}(j)}{d_{r(u)}}\right|^{p_i-1} = \sum_{j\in\Gamma_i} u(j)\left|\frac{h_{p_i}(j)}{d_{r(u)}}\right|^{p_i-1}\operatorname{sgn}(h_{p_i}(j))\\
&\quad= -\sum_{j\in\Gamma_i^c} u(j)\left|\frac{h_{p_i}(j)}{d_{r(u)}}\right|^{p_i-1}\operatorname{sgn}(h_{p_i}(j)) \leqslant \|u\|_1.
\end{aligned}
$$

Finally, for $i > N$,

$$
\begin{aligned}
&p_i|h_{p_i}(j_0) - h^*_\infty(j_0)|\\
&\quad\leqslant |u(j_0)||h_{p_i}(j_0) - h^*_\infty(j_0)| + d_{r(u)}|u(j_0)|\left(1 + (p_i - 1)\frac{|h_{p_i}(j_0) - h^*_\infty(j_0)|}{d_{r(u)}}\right)\\
&\quad\leqslant |u(j_0)|\|h_{p_i} - h^*_\infty\| + d_{r(u)}\|u\|_1 \leqslant 2|u(j_0)|\|h^*_\infty\|_1 + d_{r(u)}\|u\|_1. \qquad \square
\end{aligned}
$$

Corollary 1. *If K, h_p and h^*_∞ are given as in Theorem* (2), *then*

$$\lim_{p\to\infty} h_p = h^*_\infty.$$

Theorem 3. *If K, h_p and h^*_∞ are given as in Theorem* 2, *then $p\,\|h_p - h^*_\infty\| \to 0$ as $p \to \infty$ if and only if, for all $1 \leqslant k \leqslant r$ and every $v \in \mathcal{V}_k$,*

$$\sum_{j\in J_{\sigma(k)}} v(j)\operatorname{sgn}(h^*_\infty(j)) = 0, \tag{9}$$

where the spaces $\mathcal{V}_k$ $(1 \leqslant k \leqslant r)$ are as in Lemma 2.

Proof. ($\Rightarrow$) Let v be a vector in $\mathcal{V}_k$. Since $J_{\sigma(k)}$ is finite and $h_p \to h^*_\infty$ as $p \to \infty$, there exists N such that, for $p > N$, $\|h_p - h^*_\infty\| < \frac{1}{2}(d_{\sigma(k)} - d_{\sigma(k)+1})$ and $\operatorname{sgn}(h_p(j)) = \operatorname{sgn}(h^*_\infty(j))$ for all $j \in J_{\sigma(k)}$. Thus, if $j \in J_{\sigma(k)}$, then

$$\lim_{p\to\infty}\left|\frac{h_p(j)}{d_{\sigma(k)}}\right|^{p-1} = \lim_{p\to\infty}\left(1 + \frac{h_p(j) - h^*_\infty(j)}{h^*_\infty(j)}\right)^{p-1} = 1,$$

because $\lim_{p\to\infty} p\,(h_p(j) - h^*_\infty(j)) = 0$. On the other hand, if $\Omega_k = J_0 \cup (\cup_{l>\sigma(k)} J_l)$ and $p > N$, then

$$\|h_p\|_{\Omega_k} \leqslant \|h^*_\infty\|_{\Omega_k} + \|h_p - h^*_\infty\|_{\Omega_k} < \frac{1}{2}\left(d_{\sigma(k)} + d_{\sigma(k)+1}\right) < d_{\sigma(k)}.$$

Applying (7) to the vector v and dividing by $d_{\sigma(k)}^{p-1}$, we have

$$\sum_{j\in J_{\sigma(k)}} v(j)\left|\frac{h_p(j)}{d_{\sigma(k)}}\right|^{p-1}\operatorname{sgn}(h_\infty^*(j)) + \sum_{j\in\Omega_k} v(j)\left|\frac{h_p(j)}{d_{\sigma(k)}}\right|^{p-1}\operatorname{sgn}(h_p(j)) = 0. \quad (10)$$

So, letting $p\to\infty$ we obtain (9).

($\Leftarrow$) Suppose that $p\,\|h_p - h_\infty^*\|$ does not converge to 0 as $p\to\infty$. This is just equivalent to the existence of a sequence $p_i\to\infty$ such that $p_i\|h_{p_i} - h_\infty^*\|\to\mu>0$. Consider the vectors u_i defined as in (5) and let u be its corresponding vector limit (6). Recall that $u\notin\mathcal{V}_0$. We obtain an equation similar to (10) by applying (7), con $v=u$ and $p=p_i$, and dividing by $d_{r(u)}^{p-1}$,

$$\sum_{j\in J_{r(u)}} u(j)\left|\frac{h_{p_i}(j)}{d_{r(u)}}\right|^{p_i-1}\operatorname{sgn}(h_\infty^*(j)) + \sum_{j\in J_{r(u)}^c} u(j)|\left|\frac{h_{p_i}(j)}{d_{r(u)}}\right|^{p_i-1}\operatorname{sgn}(h_{p_i}(j)) = 0. \quad (11)$$

In this case, for $j\in J_{r(u)}$,

$$\lim_{i\to\infty} p_i(h_{p_i}(j) - h_\infty^*(j)) = \lim_{i\to\infty} p_i\|h_{p_i} - h_\infty^*\|\,\frac{h_{p_i}(j) - h_\infty^*(j)}{\|h_{p_i} - h_\infty^*\|} = \mu\,u(j).$$

So, taking limits in (11) and keeping in mind that $\alpha e^{\beta\alpha}>\alpha$ for all $\alpha\neq 0$ and $\beta>0$, we obtain,

$$\begin{aligned} 0 &= \sum_{j\in J_{r(u)}} u(j)e^{\mu u(j)/h_\infty^*(j)}\operatorname{sgn}(h_\infty^*(j)) \\ &= \sum_{j\in J_{r(u)}} u(j)\operatorname{sgn}(h_\infty^*(j))e^{(\mu/d_{r(u)})u(j)\operatorname{sgn}(h_\infty^*(j))} > \sum_{j\in J_{r(u)}} u(j)\operatorname{sgn}(h_\infty^*(j)). \end{aligned}$$

Then (9) does not hold for a vector $v\in\mathcal{V}\setminus\mathcal{V}_0$. □

Theorem 4. *If K, h_p and h_∞^* are given as in Theorem 2, then there exists $p_0>1$ such that $h_p = h_\infty^*$ for all $p>p_0$ if and only if, for all $v\in\mathcal{V}$ and all $n\in\mathbb{N}$,*

$$\sum_{j\in J_n} v(j)\operatorname{sgn}(h_\infty^*(j)) = 0. \quad (12)$$

Proof. By Theorem 1 we have $h_p = h_\infty^*$ for all $p>p_0$ if and only if

$$\sum_{j\in\mathbb{N}} v(j)|h_\infty^*(j)|^{p-1}\operatorname{sgn}(h_\infty^*(j)) = 0,$$

for all $v\in\mathcal{V}$. Since $h_\infty^*(j)=0$ for $j\in J_0$, we can write the above equation as

$$\sum_{n\in\mathbb{N}} d_n^{p-1}\sum_{j\in J_n} v(j)\operatorname{sgn}(h_\infty^*(j)) = 0. \quad (13)$$

If (12) holds then (13) follows trivially. On the other hand, if (13) is true, then dividing the last equation by d_i^{p-1}, for $i = 1, 2, \ldots$, respectively, and taking limits as $p \to \infty$ we obtain (12) by means of an inductive procedure. □

Lemma 4. *There exists $M > 0$ such that, for p large enough,*

$$\|h_p - h^*_\infty\|_{\hat{J}} \leqslant \|h_p - h^*_\infty\| \leqslant M\|h_p - h^*_\infty\|_{\hat{J}}, \tag{14}$$

where $\hat{J} = \cup_{k=1}^r J_{\sigma(k)}$.

Proof. Note that the first inequality in (14) is obvious. For the second, suppose, to the contrary, that there exists a sequence $p_i \to \infty$, with $h_{p_i} \neq h^*_\infty$, such that

$$\frac{\|h_{p_i} - h^*_\infty\|_{\hat{J}}}{\|h_{p_i} - h^*_\infty\|} \to 0 \quad \text{as } i \to \infty. \tag{15}$$

Consider for this sequence the vectors u_i as in (5) and its corresponding vector limit u. From (15) we conclude that $u(j) = 0$ for all $j \in \hat{J}$. Hence $u \in \mathcal{V}_0$. A contradiction. □

The inequalities in (14) show that the rate of convergence of $\|h_p - h^*_\infty\|$ is just determined by the set of indices $\hat{J} = \cup_{k=1}^r J_{\sigma(k)}$.

4. The main result

Let $\mathcal{W}_0 = \oplus_{k=1}^r \mathcal{V}_k$, $m_0 = \dim(\mathcal{W}_0)$, $\mathcal{B} = \{v_1, \ldots, v_{m_0}\}$ be a basis of $\mathcal{W}_0$ and $\mathbf{I} = \{1, \ldots, m_0\}$, where $\mathcal{V}_k$ are the linear subspaces of $\mathcal{V}$ given in Lemma 2. We assume that if $i \in \mathbf{I}$, then $v_i \in \mathcal{V}_k$ for some $k \in \{1, \ldots, r\}$. Let $\{I_k\}_{k=1}^r$ be the partition of $\mathbf{I}$ given be $I_k = \{i \in \mathbf{I} : v_i \in \mathcal{V}_k\}$ and put $m_k = \mathrm{card}(I_k)$.

Given any vector $v \in \mathcal{V}$ there are two unique vectors $\Lambda_v = (\lambda_v(i))_{i \in \mathbf{I}} \in \mathbb{R}^{\mathbf{I}}$ and $w_v \in \mathcal{V}_0$ such that

$$v = \sum_{i \in \mathbf{I}} \lambda_v(i) v_i + w_v.$$

Since $w_v(j) = 0$ for all $j \in \hat{J}$ (see (3)), $\|\Lambda_v\| := \max_{i \in \mathbf{I}} |\lambda_v(i)|$ is a norm on $\mathcal{V}|_{\hat{J}}$. So, by the equivalence of norms in $\mathcal{V}|_{\hat{J}}$, we have

$$M_1\|\Lambda_v\| \leqslant \|v\|_{\hat{J}} \leqslant M_2\|\Lambda_v\|, \tag{16}$$

for some constants $M_1, M_2 > 0$.

For $1 \leqslant k \leqslant r$ and $n \in \mathbb{N}$, we define

$$\Sigma(n,k) = \max_{i \in I_k} \left| \sum_{j \in J_n} v_i(j) \mathrm{sgn}(h^*_\infty(j)) \right| \text{ and } \eta(k) = \min\left\{ n \in \mathbb{N} : \Sigma(n,k) \neq 0 \right\},$$

where $\eta(k)$ is assumed to be 0 if $\Sigma(n,k)=0$, for all $n\in\mathbb{N}$. Finally, let a be the real number given by

$$a=\max_{1\leqslant k\leqslant r} d_{\eta(k)}/d_{\sigma(k)}. \tag{17}$$

Since $v_i(j)=0$ for all $j\in\cup_{l=1}^{\sigma(k)-1}J_l$ for $i\in I_k$, we have $\eta(k)\geqslant\sigma(k)$ and so $0\leqslant a\leqslant 1$.

In what follows, if A is a matrix, we will denote by A^T the transpose matrix of A and by $\|A\|$ the row-sum norm of A.

Theorem 5. *Let K be a finite-dimensional affine subspace of $\ell_1(\mathbb{N})$, $0\notin K$. Let h_p, $1<p<\infty$, denote the best ℓ_p-approximation of* 0 *from K and let h_∞^* be the strict uniform approximation of* 0 *from K. Then there are positive constants L_1, L_2 and $p_0\geqslant 1$ such that, for $p>p_0$,*

$$L_1 a^p\leqslant p\,\|h_p-h_\infty^*\|\leqslant L_2 a^p, \tag{18}$$

where a is the real number defined in (17).

Proof. If $h_p=h_\infty^*$ for all p large enough then, by Theorem 4, $\Sigma(n,k)=0$, for all n and all k. Thus $a=0$ and (18) holds. On the other hand, if $\eta(k)=\sigma(k)$ for some $k\in\{1,2,\ldots,r\}$, then $a=1$ and (18) follows from Theorems 2 and 3. Therefore, we assume $\eta(k)>\sigma(k)$, all k. This implies that $0<a<1$ and (see Theorem 3) $p\,\|h_p-h_\infty^*\|\to 0$ as $p\to\infty$.

Set $\eta=\max\{\eta(k): d_{\eta(k)}/d_{\sigma(k)}=a\}$, $\mathbf{n}=\max\{\eta,r\}$ and $\mathbf{J}=\cup_{l=1}^{\mathbf{n}}J_l$.

Let $\Lambda_p=(\lambda_p(i))_{i\in\mathbf{I}}$ be the unique vector in $\mathbb{R}^{\mathbf{I}}$ such that

$$h_p=h_\infty^*+\sum_{i\in\mathbf{I}}\lambda_p(i)v_i+w_p, \tag{19}$$

with $w_p\in\mathcal{V}_0$. Taking into account (14) and (16), we deduce that

$$\tilde{M}_1\|\Lambda_p\|\leqslant\|h_p-h_\infty^*\|\leqslant\tilde{M}_2\|\Lambda_p\|, \tag{20}$$

for some constants $\tilde{M}_1,\tilde{M}_2>0$. Hence the rate of convergence of $\|h_p-h_\infty^*\|$ will be also determined by the norm of the vector Λ_p.

Since $\mathbf{J}$ is a finite set and $h_p\to h_\infty^*$, there exists $p_0\geqslant 1$ such that for $p>p_0$,

$$2\|h_p-h_\infty^*\|<d_{\mathbf{n}}-d_{\mathbf{n}+1} \tag{21}$$

and $\operatorname{sgn}(h_p(j))=\operatorname{sgn}(h_\infty^*(j))$ for all $j\in\mathbf{J}$.

Thus for $p>p_0$, taking into account that $w_p(j)=0$ for all $j\in\mathbf{J}$ and applying the Taylor's formula of order 1 to the function $\varphi(z)=(1+z)^{p-1}$ about $z=0$, we obtain for $j\in J_l$ with $1\leqslant l\leqslant\mathbf{n}$,

$$\begin{aligned}\left|\frac{h_p(j)}{d_l}\right|^{p-1}&=\left(\frac{h_p(j)}{h_\infty^*(j)}\right)^{p-1}=\left(1+\sum_{i\in\mathbf{I}}\frac{\lambda_p(i)v_i(j)}{h_\infty^*(j)}\right)^{p-1}\\&=1+\frac{1}{h_\infty^*(j)}(p-1)\sum_{i\in\mathbf{I}}\lambda_p(i)v_i(j)+R_p(j),\end{aligned} \tag{22}$$

with $R_p(j) = o(p\|\Lambda_p\|)$ as $p \to \infty$, because $p\,\|h_p - h^*_\infty\| \to 0$ and by (20) that is just equivalent to $p\,\|\Lambda_p\| \to 0$.

Putting $v = v_i$, $i \in \mathbf{I}$, in (7) we obtain, for p large,

$$\sum_{j\in\mathbf{J}} v_i(j)|h_p(j)|^{p-1}\mathrm{sgn}(h^*_\infty(j)) + \sum_{j\in\mathbf{J}^c} v_i(j)|h_p(j)|^{p-1}\,\mathrm{sgn}\,(h_p(j)) = 0 \quad \forall i \in \mathbf{I}. \tag{23}$$

This nonlinear system can be written as

$$M\,H_p^T + K_p^T = 0, \tag{24}$$

where M is the matrix $M = (v_i(j))_{(i,j)\in\mathbf{I}\times\mathbf{J}}$ and H_p, K_p denote the vectors in $\mathbb{R}^{\mathbf{J}}$ and $\mathbb{R}^{\mathbf{I}}$, respectively, whose components are given by

$$H_p(j) = |h_p(j)|^{p-1}\,\mathrm{sgn}(h^*_\infty(j)), \quad j \in \mathbf{J}$$
$$K_p(i) = \sum_{j\in\mathbf{J}^c} v_i(j)|h_p(j)|^{p-1}\,\mathrm{sgn}(h_p(j)), \quad i \in \mathbf{I}.$$

Taking into account (22) we can express the vector H_p^T like

$$H_p^T = \Delta_{\mathbf{J}}^{p-1}\Upsilon^T + (p-1)\Delta_{\mathbf{J}}^{p-2}M^T\Lambda_p^T + \Delta_{\mathbf{J}}^{p-1}\,R_p^T,$$

where Υ and R_p are the vectors in $\mathbb{R}^{\mathbf{J}}$ given by $\Upsilon := \left(\mathrm{sgn}(h^*_\infty(j))\right)_{j\in\mathbf{J}}$ and $R_p := \left(R_p(j)\,\mathrm{sgn}(h^*_\infty(j))\right)_{j\in\mathbf{J}}$, and $\Delta_{\mathbf{J}} := (\delta_{ij})_{(i,j)\in\mathbf{J}\times\mathbf{J}}$ is the diagonal matrix such that $\delta_{jj} = d_l$ if $j \in J_l$, $1 \leqslant l \leqslant \mathbf{n}$. Substituting in (24) we obtain the system

$$M\left(\Delta_{\mathbf{J}}^{p-1}\Upsilon^T + (p-1)\Delta_{\mathbf{J}}^{p-2}M^T\Lambda_p^T + \Delta_{\mathbf{J}}^{p-2}\,R_p^T\right) + K_p^T = 0. \tag{25}$$

Let $\Delta_{\mathbf{I}} = (\tilde{\delta}_{ij})_{(i,j)\in\mathbf{I}\times\mathbf{I}}$ be the diagonal matrix such that $\tilde{\delta}_{ii} = d_{\sigma(k)}$ if $i \in I_k$, $1 \leqslant k \leqslant r$. Multiplying (25) by $\Delta_{\mathbf{I}}^{-p+2} := \left(\Delta_{\mathbf{I}}^{-1}\right)^{p-2}$ we have

$$(p-1)\Delta_{\mathbf{I}}^{-p+2}M\Delta_{\mathbf{J}}^{p-2}M^T\,\Lambda_p^T = -\Delta_{\mathbf{I}}^{-p+2}M\Delta_{\mathbf{J}}^{p-1}\Upsilon^T - \Delta_{\mathbf{I}}^{-p+2}M\Delta_{\mathbf{J}}^{p-2}R_p^T - \Delta_{\mathbf{I}}^{-p+2}K_p^T. \tag{26}$$

Observe that the multiplication by $\Delta_{\mathbf{I}}^{-p+2}$ is equivalent to divide by $d_{\sigma(k)}^{p-2}$ each of equations in (23) obtained for $i \in I_k$. This operation is justified because $v_i(j) = 0$ for all $j \in J_l$ if $j < \sigma(k)$.

Next we study each of the terms in the former system. Let us partition M into blocks $M_{k,l}$, $k = 1, \ldots, r$, $l = 1, \ldots, \mathbf{n}$, where $M_{k,l} = (v_i(j))_{(i,j)\in I_k\times J_l}$. An easy computation shows that

$$A(p) := \Delta_{\mathbf{I}}^{-p+2}M\Delta_{\mathbf{J}}^{p-2}M^T = \left(A_{k,s}(p)\right)_{k=1,\ldots,r}^{s=1,\ldots,r},$$

where $A_{k,s}(p)$ is the matrix of order $m_k \times m_s$ given by

$$A_{k,s}(p) = \sum_{l=1}^{\mathbf{n}}\left(\frac{d_l}{d_{\sigma(k)}}\right)^{p-2} M_{k,l}M_{s,l}^T.$$

Since $M_{k,l}$ is a null matrix if $l < \sigma(k)$, and $d_l < d_{\sigma(k)}$ if $l > \sigma(k)$, then

$$A_{k,s} := \lim_{p\to\infty} A_{k,s}(p) = M_{k,\sigma(k)} M^T_{s,\sigma(k)}.$$

Moreover, since $M_{s,\sigma(k)}$ is also a null matrix if $s > k$, we conclude that $A := \lim_{p\to\infty} A(p)$ is a lower triangular matrix by blocks and so

$$\det(A) = \prod_{k=1}^{r} \det\left(M_{k,\sigma(k)} M^T_{k,\sigma(k)}\right) \neq 0.$$

In particular we have proved that there exists $p_1 \geqslant p_0$ such that the matrix $A(p)$ is non singular for $p \geqslant p_1$.

Analogously, denoting by $B_p = -\Delta_{\mathbf{I}}^{-p+2} M \Delta_{\mathbf{J}}^{p-1} \Upsilon^T$ it is easy to check that

$$B_p(i) = -d_{\sigma(k)} \sum_{l=\sigma(k)}^{\mathbf{n}} \left(\frac{d_l}{d_{\sigma(k)}}\right)^{p-1} \sum_{j\in J_l} v_i(j)\,\mathrm{sgn}(h^*_\infty(j)) \quad \text{for } i \in I_k, \quad 1 \leqslant k \leqslant r.$$

The definition of a implies that if $d_l/d_{\sigma(k)} > a$ then $\sum_{j\in J_l} v_i(j)\,\mathrm{sgn}(h^*_\infty(j)) = 0$ for all $i \in I_k$. On the other hand, the selection of $\mathbf{n}$ implies that there is $k_0 \in \{1, 2, \ldots, r\}$ and $l_0 = \eta(k_0)$ such that $k_0+1 \leqslant l_0 \leqslant \mathbf{n}$, $d_{l_0}/d_{\sigma(k_0)} = a$ and $\Sigma(l_0, k_0) = \max_{i\in I_{k_0}} \left| \sum_{j\in J_{l_0}} v_i(j)\right.$ $\left.\mathrm{sgn}(h^*_\infty(j))\right| \neq 0$. Therefore,

$$0 < b := \lim_{p\to\infty} \|B_p\|/a^p < \infty.$$

Similarly, writing $C_p = -\Delta_{\mathbf{I}}^{-p+2} M \Delta_{\mathbf{J}}^{p-2} R_p^T$ we obtain, for $i \in I_k$, $1 \leqslant k \leqslant r$,

$$C_p(i) = -d_{\sigma(k)} \sum_{l=\sigma(k)}^{\mathbf{n}} \left(\frac{d_l}{d_{\sigma(k)}}\right)^{p-1} \sum_{j\in J_l} v_i(j) R_p(j)\,\mathrm{sgn}(h^*_\infty(j))$$

and then $\lim_{p\to\infty} \frac{\|C_p\|}{p\,\|\Lambda_p\|} = 0$.

Finally, denoting $D_p = -\Delta_{\mathbf{I}}^{-p+2} K_p^T$, we have

$$D_p(i) = \sum_{j\in\mathbf{J}^c} v_i(j) \left|\frac{h_p(j)}{d_{\sigma(k)}}\right|^{p-1} \mathrm{sgn}(h_p(j)) \quad \text{for } i \in I_k, \quad 1 \leqslant k \leqslant r.$$

Since, for $p > p_1$ (see (21)),

$$\|h_p\|_{\mathbf{J}^c} \leqslant \|h^*_\infty\|_{\mathbf{J}^c} + \|h_p - h^*_\infty\|_{\mathbf{J}^c} \leqslant d_{\mathbf{n}+1} + \|h_p - h^*_\infty\| < \frac{1}{2}(d_{\mathbf{n}} + d_{\mathbf{n}+1}) < d_{\mathbf{n}},$$

and, from the selection of $\mathbf{n}$, $d_{\mathbf{n}}/d_{\sigma(k)} \leqslant a$, for all $k \in \{1, 2, \ldots, r\}$, we conclude that $\lim_{p\to\infty} \|D_p\|/a^p = 0$.

With the notation introduced in the previous paragraphs we can write the system (26) as

$$(p-1)A(p)\Lambda_p^T = B_p + C_p + D_p,$$

and so

$$(p-1)\|\Lambda_p\| = \|A(p)^{-1}\left(B_p + C_p + D_p\right)\| \leqslant \|A(p)^{-1}\| \left(\|B_p\| + \|C_p\| + \|D_p\|\right).$$

Therefore,

$$(p-1)\|\Lambda_p\|\left(1 - \frac{\|A(p)^{-1}\|\|C_p\|}{(p-1)\|\Lambda_p\|}\right) \leqslant \|A(p)^{-1}\|\,\|B_p\| + \|A(p)^{-1}\|\,\|D_p\|.$$

Dividing the above inequality by a^p and taking limits as $p \to \infty$ we have $\limsup_{p\to\infty} p\|\Lambda_p\|/a^p \leqslant \|A^{-1}\|\,b$.

In similar way,

$$\|B_p\| \leqslant (p-1)\|A(p)\|\,\|\Lambda_p\|\left(1 + \frac{\|C_p\|}{(p-1)\|A(p)\|\,\|\Lambda_p\|}\right) + \|D_p\|$$

and therefore $\liminf_{p\to\infty} p\,\|\Lambda_p\|/a^p \geqslant b/\|A\|$. From the above inequalities there exists $p_2 > p_1$ such that, for $p > p_2$,

$$\frac{b}{\|A\|} \leqslant \frac{p\,\|\Lambda_p\|}{a^p} \leqslant b\,\|A^{-1}\|. \tag{27}$$

Finally, taking into account (20) we conclude the proof. □

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